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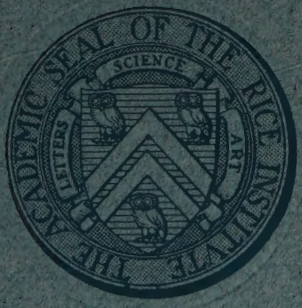

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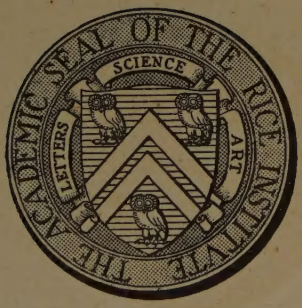

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THE EARLY SCIENTIFIC WORK OF HENRI POINCARÉ [1]

SPEAKING of the early years of the work of Poincaré is a more ambitious task than the title alone would suggest. One of the most remarkable features of that work is, indeed, the number and importance of the results which appeared almost simultaneously in the short period between the years 1879 and 1884-5, changing entirely the state of mathematical science and opening new paths for it in all of its directions. So rapid was the outburst of these Memoirs devoted to the most varied branches of mathematics that volume XI of the *Bulletin de la Société Mathématique de France,* for instance, contains three which are almost immediately consecutive (two of them having been presented at the same fortnightly meeting of the Society) and each of them constituting an entirely new chapter in the Theory of Functions.

The ideas which germinated with such rapidity often seemed to precede by a quarter or a half a century the time when they were used. The ways thus opened were not always immediately followed, as if science required time to keep pace with such an advance. But now, after that time has elapsed, they have become the very basis of almost all contemporary works and improvements in Analysis. We shall eventually try, when reviewing them, to give an idea of the further development and continuation of some of them.

[1] Lectures delivered at the Rice Institute in March, 1920, by Professor J. Hadamard, of the Collège de France and the Ecole Polytechnique.

The main, and central, subject of modern Analysis lies in the two connected theories of Functions and of Differential Equations, ordinary or partial. This double subject is already attacked in Poincaré's first mathematical work—we mean his thesis for the Doctorate.

After Cauchy had proved his fundamental theorem on differential equations under *regular* initial conditions, the study of *singular* ones was begun by Briot and Bouquet. One of their chief results concerns, as is well known, the case of *one* differential equation when the expression of the differential coefficient appears in the form 0/0. It is known that there is no difficulty in obtaining analytic solutions except in the case of a certain coefficient λ being a positive integer.

Poincaré's thesis attacks the corresponding problem for a system, the system being of the form

$$(1) \qquad \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt$$

and all the X's vanishing for $x_1 = x_2 = \ldots = x_n = 0$, in which case, exceptional subcases being excepted, we can admit, by means of classic linear transformations on the variables, that we have

$$x_i = \lambda_i x_i + \ldots \qquad (i = 1, 2, \ldots, n)$$

where dots stand for terms of higher degree and the λ_i's are properly chosen constants. More precisely, instead of the system (1), Poincaré considers the corresponding linear partial differential equation

$$(1A) \qquad X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \ldots + X_n \frac{\partial f}{\partial x_n} = 0,$$

the integration of which is fully equivalent to the integration of (1). Such a problem would seem, at first sight,

to be an obvious generalization of that of Briot and Bouquet. But this is by no means the case: as will occur in most of Poincaré's subsequent researches, the difficulty of the question is shown by the new and remarkable circumstances appearing in the solution. Indeed, the conditions imposed on the λ_i's, for the existence of the solutions, profoundly differ from those which were necessary for the single parameter λ of Briot and Bouquet. They are of two types:

(a) Certain linear arithmetical relations between these λ_i's are excluded, this being analogous to Briot and Bouquet's condition for λ;

(b) In the plane for the complex λ-variable, the origin must lie outside the smallest convex polygon which contains the points representative of the λ_i's.

The latter condition is of quite a new nature, and it has led Poincaré to very simple forms, unknown at that epoch, for analytic functions with *lacunary spaces*. It is clear indeed, that, if $n - 1$ of the λ's are given, the remaining one is bound, on account of (b), to lie outside a certain polygonal region. The latter, therefore, constitutes a space of non-existence for the solutions of (1A), or more exactly, of other neighboring partial differential equations to which the integration of (1A) is referred, when these solutions are considered as functions not of the x's but of the λ's.

This fact, to which we shall return later, shows us the essential difficulty of this first problem treated by Poincaré. Moreover, another relation with the Theory of Functions appears with the notion of what Poincaré calls "algebroid" functions of several variables and a corresponding important theorem [1] which has been known in Germany as the Vorbereitungssatz of Weierstrass (also called the Factor-

[1] See Osgood's Lecture IV in the Madison Colloquium, 1913.

ization Theorem), as it seems to have been given by Weierstrass in one of his courses at the University of Berlin in 1860. As to printed publication, priority would undoubtedly belong to Poincaré if it were not that the first author is Cauchy:[1] the coincidence being evidently due to the importance of the theorem in question, which, indeed, has played an essential part in Poincaré's further researches in functions of several variables.

On the other hand, the main results concerning (1) and (1A), as well as algebroid functions, did not remain useless in Poincaré's later work. The expressions of the celebrated "asymptotic solutions," in Analytical and Celestial Mechanics, depend on these principles.

*
* *

During the time in which he was engaged in solving this first question in Integral Calculus, Poincaré was also devoting himself to Arithmetic, under the influence of another of his senior contemporaries, the great French mathematician, Hermite.

However closely related with Hermite's work, these papers of Poincaré's curiously show the differences between these two great minds. While Hermite's methods, to my mind, at least, generally keep an unexpected and somewhat mysterious character, we always feel, in Poincaré's ideas, a peculiar clearness and simplicity which make them appear like "Columbus' egg." It is the more important to mention these first arithmetical works of Poincaré because, having been issued before the time of his universal fame, they seem to have been most frequently overlooked, and new studies of them would be of interest.

[1] Bulletin de Ferussac, 1831. "Exercises d'Analyse," vol. II. Cauchy's priority was discovered by Lindelöf "Leçons sur la Théorie des résidus." Note of p. 27.

One of the most important and beautiful ideas of Hermite concerns, as is well known, the reduction of indefinite quadratic forms; and it is also known that his success in the reduction of such a form as $(x + \alpha y)\ (x + \beta y)$ is due to the introduction of the auxiliary form $(x + \alpha y)^2 + \lambda^2 (x + \beta y)^2$, which is a definite one and can be reduced by classic methods, for every determinate value of the parameter λ: this being done, Hermite applies to the former expression the same substitution which has succeeded in reducing the latter.

The method has been generalized, by Hermite himself and others, especially Jordan, to higher degrees and higher numbers of variables. Nevertheless, there remains something puzzling in it, and there seems *a priori* to be no reason why the substitution which has reduced one form should be precisely the one which is fit to reduce another one.

This, on the contrary, becomes clear in Poincaré's presentation, namely, by a comparison of the arithmetic reduction with the algebraic one, and, on the other hand, by transferring the notion of reduction from forms to substitutions. Algebraically speaking, the given form can be reduced to a canonical type, in the above example, XY, by a certain linear substitution S. Now, as S does not generally belong to the "arithmetical group"—that is, has not integral numbers for its coefficients—it has, according to Poincaré, to be "reduced with respect to this group," that is, decomposed in a unique way, into a product $S_0 T$, where T belongs to the arithmetical group and S_0 satisfies properly chosen conditions C, which are such that no two substitutions S_0 and S_0' satisfying C can give a quotient $S_0' S_0^{-1}$ belonging to the arithmetical group: which reduction of S is performed by considering the effect of S on a canonical

definite form. The intimate reason of Hermite's device thus becomes perfectly evident.

The same clearness and simplicity was given to the theory of ideals in the quadratic case—which had just been established on solid bases by the fundamental Memoirs of Dedekind—when Poincaré [1] considered it from the geometric point of view with the help of Bravais' notion of nets [2]; and, moreover, the relation of this theory with the arithmetical theory of binary quadratic forms is set in evidence. All this sensibly coincides with what was found again, years later, in 1893, by Klein, as may be confirmed by reading the eighth lecture of his Evanston Colloquium.

In both the above cases, there was only new interpretation of known results. Wide extensions lie before us when we follow Poincaré in the concepts of *order* and *genus*.[3]

These notions, in so far as binary quadratic forms are concerned, date from the work of Gauss. But the way in which they are introduced by him is quite special to this particular case, and does not seem to admit of any generalization. Let me remind you that in his presentation the notion of genus rests on the quadratic character of numbers, or the corresponding symbol of Legendre. The determinant D having been decomposed into prime factors and the odd prime p being one of these factors, the given form is able only to represent residues, or only to represent non-residues with respect to p: which criterion, being successively applied to every prime factor of D, gives a classification of the various forms—or rather classes of forms,—

[1] Journal de l'Ecole Polytechnique, vol. XXVIII, 1880.

[2] I must confess that, before reading Poincaré's Memoir, I was not aware of the greatness of Bravais' work, which deals with most various and important parts of Science, and, especially, contains some important initial results found again later on by Minkowski.

[3] Comptes Rendus de l'Académie des Sciences, Paris, January 9-16, 1882.

with D for their determinant, into several categories or "genera." Such an argument is quite special to the binary quadratic case; moreover, there is, as usual, the difficulty concerning the modulus 2, and the connection between treatment for the latter number and for odd primes is not indicated by any general idea.

Now such an idea lies at the basis of Poincaré's analysis, and a simpler one could hardly be thought of. Instead of considering the ordinary equivalence of two forms, which leads to the distribution of forms into classes, we introduce the notion of *equivalence with respect to a given modulus* M. For any degree and number of variables, two forms, f and g, are said to be equivalent with respect to the modulus M if there is a linear substitution belonging to the arithmetical group which transforms f into a form f_1 such that $f_1 \equiv g \pmod{M}$.

This definition being given, the definition of genus immediately follows, namely, two forms are said to belong to the same genus if they are equivalent with respect to *every* integral modulus M.

How this leads to the ordinary notion of genus for Gauss' case, is to be seen by treating, more generally, quadratic forms in n variables. It appears that any two forms of this kind with the same determinant D are *always* equivalent with respect to every modulus prime to $2D$. Thus, powers of prime factors of D and powers of 2 are the only moduli which call for special treatment.

For the above-mentioned reasons, this theory of Poincaré does not seem to me to have awakened as much attention as it deserves, and would certainly lead to further interesting studies. Problems corresponding to those which have been classically treated concerning genera of binary quadratic forms, number of actually possible genera, num-

ber of classes in each genus, etc., deserve to be studied for quadratic forms in n variables; and as far as I know, the theory has not been applied to higher degrees since Poincaré.[1]

But when the two notes just mentioned appeared, the time had come when Poincaré's name was going to become universally known throughout the scientific world: it was the time when the fuchsian functions, in Humbert's phrase, "burst out."

[1] Poincaré himself takes the first steps for the study concerning cubic forms.

FUCHSIAN FUNCTIONS

CAUCHY'S Theory of Functions had found in the case of elliptic functions the best opportunity of strikingly showing its fecundity and power. Let us recall the main features of this particular application of Cauchy's theory.

The first element of success, in this case, consists in having introduced, instead of the multiform elliptic integral of the first kind

$$(2) \quad u = \int_{\infty}^{z} \frac{dz}{\sqrt{R(z)}} \qquad (R(z) = 4z^3 - g_2 z - g_3),$$

the inverse function $p(u)$, defined by the condition that (2) is equivalent to

$$(2A) \qquad z = p(u)$$

and in the fact that such an inverse function is *uniform*. It must not, however, be forgotten that such a fact is necessarily connected with the choice of the integral of the *first* kind. Inverse functions of elliptic integrals of the second or third kind are not uniform.

Furthermore, the basis of all the properties of such a uniform function (2A) lies in its double periodicity, which may be expressed by saying that (2A) remains invariant by the change of u to any of the quantities

$$(2B) \quad u + 2m\omega + 2m'\omega' \qquad (m, m' = 0, \pm 1, \pm 2, \ldots)$$

that is, by ∞ substitutions which form a group.

Finally, everything is brought to the utmost simplicity by a geometric consideration, namely, the introduction of the *parallelogram of periods*, which is for the group (2B)

a *reduced domain,* this meaning that any point of the plane has one and only one homologue within that parallelogram.

The theory of elliptic functions admits of a well-known generalization—the theory of abelian functions—the group (2B) being replaced by an analogous one in several variables. But the more beautiful and harmonious these two theories had proved to be, the less inclined the geometers were to suppose that they could be extended to other than additive groups.

Now that this extension had been obtained, we wonder at the fact that mathematicians and even at first Poincaré himself, should have thought it to be impossible. For a first example of a non-additive group with a uniform invariant function was known since the works of Hermite on the modular function: the group in this case was formed with linear fractional, or homographic, substitutions

$$\frac{\alpha u + \beta}{\gamma u + \delta} \tag{2C}$$

where α, β, γ, δ were, in this case, ordinary integers with $\alpha\delta - \beta\gamma = 1$.

This first indication was not unique. Two other memoirs, by Schottky and Schwarz, had given similar ones. Schottky's memoir, which, though fundamental even from other points of view than our present one, had received less notice than it deserved and was unknown to Poincaré when beginning to work on the subject, already introduced some rather general and complicated categories of what were to be "fuchsian groups," because of the invariance of certain transcendental functions. Schwarz's subject was more special, but generalized from a certain point of view Hermite's researches: it dealt with Gauss' hypergeometric series and the corresponding linear differential equation.

Even the connection between this problem of Integral Calculus and a group of the kind (2C) was, in all these cases, already apparent, by virtue of the introduction of the *ratio* $u = y_1 : y_2$ of two independent integrals of a linear differential equation of the second order with rational coefficients defining y as a function of a (so far, independent) variable z: such a ratio, indeed, undergoes a linear fractional substitution each time z circulates around singular points. Moreover, a reason—though not a sufficient one, as will be shown by Poincaré's researches—why the inverse function $z(u)$ may chance to be a uniform one already appeared on account of the well-known identity connecting two independent solutions y_1, y_2 and their derivatives.

Now, in spite of all these clues, geometers had no idea of anything like what was going to be revealed to them; and we know, from Poincaré [1] himself, that his first idea was to prove that Hermite's example could *not* be generalized; but he also tells us at the same time how the contrary was shown to him by one of those spontaneous and seemingly unprepared intuitions which, in his mind—if not in every mathematician's—lay at the base of discovery.

Poincaré will operate under conditions incomparably more general and more various than the creators of the Theory of Elliptic Functions; but the successive steps will be quite analogous in both cases.

Instead of the problem of quadrature defined by (2), he will consider a linear differential equation of any order with algebraic coefficients. Though the latter problem stands on a considerably higher plane of difficulty than

[1] See "Science et Méthode," p. 43-63. In this highly interesting paper, reproducing a lecture delivered before the Psychological Society of Paris, Poincaré describes the successive steps through which discovery developed in his own mind; and the example chosen is precisely the case of fuchsian functions.

the former, linear differential equations offer themselves as the simplest of all. To the multiform character of the function (2), corresponds the multiform character of solutions of linear differential equations, which was known from the fundamental works of Fuchs, whose name will, therefore, be given to the new conception. Therefore also, the rôle of the periodicity of (2A) will be played by a certain *group* of linear substitutions. In the case of an equation of the second order, which will correspond to the first step of the method, we have seen that we can deal with a group of homographic substitutions in one variable, that variable being the ratio u of two solutions of the equation. But in order that z may be a uniform function of u, we may not deal with *any* linear fractional group: the group must be *discontinuous,* namely, the different values of u deduced from each other by all the substitutions of the group must be, at least generally, isolated, each of them being the centre of a circle within which none of the others is contained. This property may cease at certain particular points, isolated or not, which may be called singular points of the group and will necessarily be essential singularities of the corresponding invariant function $z(u)$. Such was the case for the modular function, the group of which was the arithmetical one defined above. It can be shown directly that the group is discontinuous; but, as Poincaré remarks, the same fact results, without any new proof, from the very existence of the corresponding uniform invariant function, therefore from Hermite's results. There is, in this case, a whole line of singular points, namely, the real axis.

Just as in the case of elliptic functions, we shall, in the first place, consider not the functions $u(z)$ but the inverse function $z(u)$, and, in turn, begin not with this function,

but by the study of the group. But in one case as in the other, such a simplification is not yet sufficient and only becomes so by reducing everything to geometric considerations. The analogue of the parallelogram of periods is what Poincaré calls the "polygone générateur," and which is the reduced domain within which any point of the plane —or, at least, of the region of existence of $z(u)$—can be brought, in a generally univocal way, by a substitution of the group. Such a reduced domain had been discovered in each of the special cases of Hermite, Schwarz, Schottky; but Poincaré pointed out that its existence is a necessary fact for any discontinuous group.[1] This being acquired, such a polygon (the sides of which are properly chosen circular arcs or, more exactly, can be assumed so without diminishing generality) fully deserves the name given to it by Poincaré: it truly "generates" the group, all the substitutions and all the properties of which can be easily deduced from it, when a definite law of correspondence between its sides has been specified. Thus the whole theory, just as happened with elliptic functions, will rest on geometric principles and will be penetrated by them in such a way that they essentially occur in every step of it: once more showing how false is the distinction between geometric and analytic researches, between geometric and analytic minds—a distinction which was even offered, we see how absurdly, in criticism of Poincaré.

A preliminary question arises in this line, however: for not any polygon P_0 with circular sides is fit to generate thus a discontinuous group. This is only the case when, on transforming the polygon P_0 into neighboring ones—which

[1] This was a kind of intuition in his first Memoir on the subject, but becomes perfectly rigorous, with the indication of the rule which is now universally known, in one of the later Memoirs (Acta Mathematica, vol. IV).

is done by means of perfectly determined substitutions—and repeating analogous operations indefinitely, these successive polygons never happen to overlap and a certain region R of the plane, the region of existence of the group and therefore of the invariant function, is thus covered, once and once only.

The first part of this question is easily solved by Poincaré by considering, in the first place, those homologues of P_0 which have one vertex common with it, the condition for no overlapping appearing then without any difficulty: the question being just solved for the immediate neighborhood of each vertex of P_0, it can be shown that the same conclusion holds throughout the whole region R, provided it be simply connected.

It remains to determine the form of the region R, which is more difficult, and here a capital distinction occurs.

Fuchsian groups, which Poincaré initially considers exclusively, are not *any* discontinuous homographic groups: on the contrary, they are distinguished amongst these by the supplementary condition of being either real, or more generally, of leaving invariant a certain circular region, the so-called "fundamental circle," the latter case being equivalent to the former, in which the circumference of the fundamental circle reduces to the real axis, by means of an obvious transformation.

In this case, Poincaré shows that the region R has no other boundary than the aforesaid straight line or circle, so that it consists, in the former case, of the whole upper half plane $y \gg 0$, where $z = x + yi$, and, in the latter, of the whole inside of the circle. He succeeds in doing this by introducing a remarkable interpretation of Lobatchewsky's Geometry, the consideration of which, as he has told

us,[1] was one of the steps which had led him to his discovery of fuchsian functions, and it is by virtue of this proof that the theory of fuchsian groups is fully constituted.

We ought, now, to enter into the study of the corresponding invariant functions, namely, fuchsian functions. But we must not yet leave geometric considerations, inasmuch as we have still more to learn from them. We have already spoken of the points where the property of discontinuity of the group disappears, namely, around which homologues of any determinate point inside the fundamental circle accumulate in infinite number. We know that such points, the singular points of the group, which necessarily exist as soon as the substitutions are infinite in number and which, on account of the argument just mentioned, necessarily lie on the circumference of the fundamental circle, must also be singular for the invariant functions. In Hermite's case, such singularities covered the whole of the circumference. This unique example given by Hermite thus appears to us as a very special subcase of the extensive category given by certain families of fuchsian groups and functions. It must be remembered that at that time, singular lines were not as familiar to analysts as they now are, and Hermite's example was almost the only one known.

The opposite case, where singular points do not fill the whole circumference of the fundamental circle, and the invariant functions can, therefore, be extended outside it, seemed at first to be much simpler and much more analogous to what was previously known. On the contrary, it is the newer and more remarkable of the two. The configuration constituted by these singular points is indeed noth-

[1] See the above quoted Lecture in "Science et Méthode."

ing else than the *perfect discontinuous set,* one of the most celebrated conquests of the general theory of point sets.

Now, the latter was still unknown when Poincaré's Memoir was published. Only later on, these paradoxical sets were found a second time by Bendixson and Cantor. They occurred to Poincaré, as a consequence of his principles, before being built up artificially by the creators of the theory of sets.

But other, and still more unexpected, geometric phenomena again appear when we leave the fuchsian case for the opposite one where no fundamental (that is to say, invariant) circle exists—the case of *kleinean groups* according to Poincaré's terminology. These "kleinean" groups may also admit of singular lines L, and the latter are no longer circles. But, losing that simple form, the lines L necessarily acquire one of those forms which lay beyond the reach of older Mathematics, which we should not be able to have any idea of without the help of modern Analysis. These curves bear the name of Jordan. Ordinary Differential Calculus cannot be applied to them: at some of their points they have a tangent, but no osculating circle; at others, not even the tangent exists.

Examples of tangentless curves were known of course, since Riemann and Weierstrass; but everyone will understand what a different thing it is to have constructed such "teratologic cases" artificially, with the purpose of obtaining them, or to meet with them in the natural development of the theory of one of the most essential and well-known problems of Integral Calculus; and this had not been the case before Poincaré.[1] Such consequences most evidently

[1] Only one other example has been found even to-day, as far as I know: it occurs in the problem of iteration, as was discovered by Fatou, Julia and Lattès when the problem was proposed by the French Academy of Sciences.

show how deeply the methods used penetrate into the intimate nature of things.

Is this geometric study exhausted? I do not think so. Klein, who had immediately become deeply interested in Poincaré's first work on this theory and had even, at the same time as Poincaré, obtained some further parts of it, has already brought its exposition to a relatively high degree of perfection in his well-known treatise on automorphic functions.[1] In particular, the reader will find there an accurate study of the line L mentioned above: the authors show that points on such a curve are individuated by a kind of decimal expansion with a properly chosen base of numeration so that a correspondence could be established between them and the values of a parameter t, according to Jordan's definition. For some of these points, the question of the existence of the tangent is solved, but not for all, especially not for those which correspond to irrational values of t, the corresponding fractional expansion being, accordingly, neither finite nor periodic; and my opinion is that such a concrete example of a Jordan curve would give place to further precise and interesting studies.

Fuchsian groups, on the other hand, are related to differential equations, considered from Poincaré's point of view, as we shall see further on. This is what I experienced when I tried to apply Poincaré's methods to geodesics of surfaces, or, more generally, varieties with negative curvature. The non-euclidean plane is precisely, as is well known, such a variety, on which geodesics are simply non-euclidean straight lines. Their study would, therefore, offer no special feature if considered in the ordinary way. But, in that non-euclidean plane, represented in the classical

[1] "Vorlesungen über die Theorie der automorphen Funktionen," by Robert Fricke and Felix Klein.

manner by the inside of a certain circle, let us take any given fuchsian group and agree that any two points will be considered as identical to each other if homologous with respect to the group. By this, the plane will be transformed into a variety V which may assume different forms according to the choice of the group, and the geodesics of which will correspond to our straight lines considered from the point of view just mentioned. If the generating polygon P_0 is symmetrical, that is, is divided by a certain non-euclidean straight line into two parts P_0', P_0'', symmetrical to each other, such geodesics may be interpreted as giving the motion of a material point movable within P_0' and undergoing successive reflections on its sides, a kind of billiards in which, however, reflection will take place in the non-euclidean way.

Now, the ds^2 of such a variety is of negative curvature and therefore, the corresponding geodesic problem follows the laws which I have had the opportunity of investigating in the *Journal de Mathématiques pures et appliquées* (1897). Especially, the occurrence of perfect discontinuous sets, as seen above, may be considered as a special case of a similar phenomenon concerning varieties of negative curvature in general.

But, nevertheless, many researches on the latter general problem remain possible, and for them the data which we possess on the special example of the above varieties V might be very useful and give important hints. For instance, it follows from researches of Professor Birkhoff that, contrary to possible inferences from the classic results of Poincaré, an indefinite approximation to the trajectories in Analytic and Celestial Dynamics, I mean such trajectories as remain at a finite distance, must not be sought for in periodic motions, but in a more general category which

Birkhoff calls "recurrent motions." The above special case of fuchsian varieties, in which a kind of decimal representation analogous to Klein's could probably be used, would evidently commend itself for the investigation of such recurrent motions and the solution of the question of the aforesaid indefinite approximation.[1]

I am not willing to leave this geometric part of the subject without mentioning a note published, in the *Bulletin de la Société Mathématique de France*,[2] a little later than those we have been considering, and which, again, seems to have been hardly noticed. It concerns non-euclidean geometries. This time, Poincaré represents them on a properly chosen quadric surface, which is an elliptic paraboloid in the case of the ordinary euclidean geometry; and, in this way, he is led to introduce not only one, but several kinds of such geometries, some of which relate to indefinite linear elements. Those evidently lead to Lorentz-Einstein theories; but even before the latter appeared, the author of the present paper had to consider such "hyperbolic" geometries when studying a problem on Riemann and Hugoniot's theory of aerial pipes.[3]

Fuchsian groups being thus determined, it remains to construct invariant functions for each of them. Here again, Poincaré's fundamental idea will be of striking simplicity.

As is well known, two kinds of methods are used in order to calculate elliptic functions, that is to say, invariant functions for the group (2B). If we follow Jacobi's ideas, we begin by constructing *relative* invariants, as is done most

[1] After having delivered these lectures, I was told by Professor Birkhoff that some young geometers around him were working in that line; but I have had no further news of their work since that time (added in 1923).

[2] Vol. XV, 1887.

[3] See my "Leçons sur la propagation des ondes," Chap. IV, p. 220-224.

commonly in researches on invariants, whether algebraic or differential; the invariants of Jacobi's are the theta functions. The quotient of two of these gives the true, or absolute, invariant.

In Weierstrass' method, the absolute invariant is reached at once by the series

$$\Sigma H(u_1), \tag{2D}$$

where H is a uniform function of u, and the u_1's denote the different quantities (2B), including u itself. The above series converges for $H(u) = 1/u^3$ and evidently gives one of the required invariants—here, the derivative of Weierstrass' p-function—because any substitution of the group only permutes the terms among themselves.

Such a method would evidently apply to other groups if we could make the series (2D) a convergent one by a proper choice of H. This is not the case for fuchsian groups. Poincaré will make up for this inconvenience by introducing, instead of (2D), the series

$$\Sigma H(u_1)\left(\frac{du_1}{du}\right)^m \tag{2D'}$$

where the successive u_1's are again assumed to be expressed as, in this case, homographic, functions of u, and du_1/du denotes the derivative of such a function. The series being assumed to be convergent, which will be the case for $m > 2$, let us apply to u any substitution of the group changing it, for example, to u_1, the u_1's will again be simply permuted among themselves, and (2D′) therefore be changed to

$$\Sigma H(u_1)\left(\frac{du_1}{du_1}\right)^m,$$

that is, multiplied by $(du/du_1)^m$, whatever the function H may be. Hence (2D′) is a relative invariant. Though

similar in its construction to Weierstrass' function, it behaves, functionally speaking, like Jacobi's symbols. Poincaré calls it a thetafuchsian function.

The quotient of two theta functions with the same m, but different H, gives the required invariant.

Fuchsian functions are constructed.

Such a wide generalization of elliptic functions preserves almost all their chief properties. The most important one is the existence of an algebraic relation between any two fuchsian functions belonging to the same group. Such a relation occurred for elliptic functions, but was a quite special one, being necessarily of the genus zero or one.

By the help of fuchsian functions, *any algebraic relation between two variables x and y can be reached:* there always exists a third variable z, of which x and y are fuchsian functions.

Moreover, *linear differential equations with coefficients rational in x and y are also integrated by means of fuchsian functions:* or, more exactly, any such differential equation (E) can be connected with a properly chosen special one (e), playing the same part as integrals of the first kind in the theory of elliptic functions, which is integrable in that way; after which the integration of (E) follows by means of new transcendentals, namely, "zetafuchsian" functions, just as elliptic integrals of the second or third kind are calculated in terms of zeta functions when the proper variable has been introduced by means of the integral of the first kind.

This grand result, the most important in the whole theory and one of the most striking in modern Analysis, is also the most difficult to prove. Poincaré succeeded in the construction of the required special equation (e) by an

audacious "method of continuity," while Klein, nearly at the same time, presented a virtually equivalent argument. There is an objection, however, which the latter did not notice and which does not constitute one of the least difficulties of the proof. The method consists in establishing a univocal correspondence between two varieties, and Poincaré saw that a special discussion is necessary as regards their boundaries.

Such a discussion achieves the proof of that theorem [1] which, as Humbert says, "brings us the keys of the algebraic world," by enabling us to study arbitrary algebraic curves as directly as we do unicursal ones with the help of rational functions, or cubics with the help of $p(u)$.

We have a well-known exposition of fuchsian functions in the aforementioned book [2] of Klein and Fricke. We refer to it especially in regard to the arithmetic properties of fuchsian functions, namely, their relations with quadratic forms, which was another striking discovery made by Poincaré at the very beginning of his studies [3]; and also in regard to transformation problems, to which important studies, since the Memoirs of Poincaré, have been devoted by several geometers. Both questions are connected by virtue of relations between a fuchsian group and its subgroups, as well as between the corresponding functions. It is unhappily doubtful whether, by this way or any other

[1] Another method resting on the solution of a boundary problem for $\nabla^2 u = e^u$, was suggested for the proof of the same theorem by Schwarz, and carried out by Picard and Poincaré himself.

[2] One objection should be mentioned. Klein deliberately avoids Poincaré's series (2D'); he has then, to "construct" the required functions another way, namely, he proves their existence with the help of the properties of conformal representation. For my part, I cannot accept such a view as replacing Poincaré's. The problem is "less solved," and considerably less in one case than in the other; to ascertain that a quantity exists is one thing; to find an explicit and even simple expression for it is quite another thing.

[3] It was even one of the steps of his discovery. See "Science et Méthode," loc. cit.

one, the result corresponding to Jacobi's on elliptic functions can be obtained: I mean replacing the slowly convergent series (2D′) by more rapidly convergent ones.

Studies of algebraic curves, of genera higher than 1, have already been carried out with the aid of their representation by fuchsian functions. In this way, Humbert found the explicit expression of abelian integrals in rational or logarithmic terms, the existence of which results from Abel's theorem; and a much more difficult problem has been solved by Poincaré himself by showing that quite special relations exist between the periods of abelian integrals when the corresponding algebraic curve satisfies a linear differential equation.[1] The latter discovery shows how hidden facts can be discovered with the help of the above mentioned principle: and even if the same facts are to be proved later on in a direct way, as we think that the case must be finally,[2] the fruitfulness of the method is none the less evident.

I do not happen to know, moreover, whether every important side of the theory of algebraic curves has been connected with fuchsian representation: I have in mind particularly Weierstrass' gap theorem and the corresponding Weierstrassian singularities, which it would be the more interesting to study from this point of view, because they represent facts invariant with respect to birational transformations. However, papers on this subject may have escaped my reading.

I may perhaps express another view, when recalling that

[1] Such an equation governs y as a function of x, that is, the intersection of the curve with an arbitrary straight line $x = \text{const.}$ A generalization could perhaps be obtained when cutting the given curve by variable curves.

[2] The author tried to do so for the two instances mentioned in the text, Acta Mathematica, 1896, and Comptes Rendus des Séances de la Société Mathématique de France, 1919, p. 46. The same is done for one of them in Appell and Goursat's "Théorie des fonctions algébriques et de leurs intégrales."

two different kinds of representations of algebraic domains have been given, namely, by fuchsian functions and by p-fold theta functions. I wonder whether there is no relation, or even no successive intermediaries, between these two points of view. A young French geometer has recently undertaken some researches which might lead in this direction.

Coming back to Poincaré's construction of fuchsian functions, we must not forget that the method is entirely general and applies to any group the substitutions of which are defined by uniform functions. Even its generalization to several variables is immediate, the derivative du_1/du being simply replaced by a Jacobian. Such cases are well known since Picard, who introduced hyperfuchsian and hyperabelian groups and functions; and an important class of groups of birational substitutions has been studied in the same way by G. Giraud.[1] No doubt other analogous extensions leading to interesting results can be found.

On the other hand, several important works have been devoted, since Poincaré, to the method of continuity or to some method by which it can be replaced, and, though passing over the solution based upon the integration of $\nabla^2 u = e^u$, connected, as is well known, with the names of Schwarz, Poincaré himself, and Picard, we should have to mention, for example, certain notes of Brouwer and several memoirs of Koebe. But the subject is undoubtedly to be resumed, not only as regards the *actual* construction of the required solution, for which a first method is given in Poincaré's Memoir, but also with the object of working out more precisely some concrete cases of a theory which is in

[1] Giraud, "Sur une classe de groupes discontinus de transformations quadratiques et sur les fonctions de trois variables indépendantes restant invariables par ces transformations," Annales Scientifiques de l'École Normale Superieure, 3-ième série, t. 32, 1915, p. 237-403.

danger of remaining "locked up in its extreme generality," as one of our geometers has said. Such studies as Hilbert's, Klein's and Smirnoff's [1] remain very useful and ought to be imitated and extended.[2]

[1] Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Leipzig, 1912, Chap. XX, p. 258 ff. Klein, Mathematische Annalen, vol. LXIV, p. 175 ff. (See also Göttinger Nachrichten, 1890.) Smirnoff, Bulletin des Sciences Mathématiques, vol. XLV, 1921, p. 93 ff.

[2] See also Klein's Suggestions in Mathematische Annalen, vol. LXIV, p. 186-7.

GENERAL THEORY OF ANALYTIC FUNCTIONS

THIS creation of fuchsian functions is the one which became the most promptly famous. It is the one that Poincaré himself treats in his most extensive memoirs, at least during that period. On the other hand, other results dating from the same period were expressed with the impressive brevity characteristic of one whom the mass of his discoveries drives on with barely time to take leave of one subject for another. For him who looks at them after the lapse of years, these other discoveries prove to be at least as effective as was the outstanding original one on the development of science. This most especially applies to the results by which Poincaré, after having introduced the above mentioned new class of functions, transforms our knowledge of analytic functions in general and analytic functions already known.

Let us recall the state of the first question, after the works of Lagrange, Cauchy, Riemann, Weierstrass, and also Méray.

The simplest of all analytic functions after polynomials, and the most analogous to polynomials, are the *integral functions,* which are holomorphic for every system of values of the variables, and can be expanded in an integral series converging everywhere. In the case which we shall examine in the first place, namely, functions of a single variable, Weierstrass had extended to them the decomposition into primary factors, just as happens for polynomials, except that suitable exponential factors are introduced.

Then we have the *meromorphic functions,* analogous to rational fractions and behaving like them in the neighborhood of every determinate point at a finite distance. Thanks to the decomposition of integral functions into factors, Weierstrass proved that such meromorphic functions in one variable can be expressed as quotients of holomorphic ones. On the other hand, Mittag-Leffler succeeded in extending to such functions the decomposition into simple elements.

These two cases are the most elementary ones; much more difficult ones may occur even for uniform functions. But these again are very special. The greatest difficulty in the theory is precisely the existence of non-uniform functions as to which, from a certain point of view, the very definition of the notion of function is defective. With respect to them, there existed nothing more than a purely negative notion. Only special classes had been considered, such as algebraic functions.

We knew hardly more on functions of several variables: in fact, all that we were acquainted with was the expansion of such an analytic function in the neighborhood of a regular point. Especially, as decomposition into factors was no longer possible, the proof of the theorem of Weierstrass concerning meromorphic functions no longer applied, and no other one was known: in other words, our knowledge of the subject was non-existent.

Such a state of science was improved by Poincaré as concerns every part of the theory. Let us summarize each of them.

Three rapid and almost telegraphic Notes in 1883 in volume XI of the *Bulletin de la Société Mathématique de France* open as many entirely new branches in the Theory

of Functions. This is, above all, the case for the Note "Sur les fonctions entières," which, though but nine pages in length, has been so rich in consequences that it has not ceased to engage the attention of geometers throughout these last thirty years. The starting point is Weierstrass' formula of decomposition into factors

$$f(x) = e^{G(x)}\Pi\left[\left(1-\frac{x}{a}\right)e^{Q_p\left(\frac{x}{a}\right)}\right], \tag{3}$$

$$Q_p(u) = u + \frac{u^2}{2} + \ldots + \frac{u^p}{p},$$

and the special case, individuated by a genial intuition of Laguerre, where the function has a *finite genus,* the number p on the right-hand side of (3) being then independent of n and $G(x)$ itself reducing to a polynomial having at most p for its degree. A new and wide question was then set by Poincaré: how is this genus connected with other properties of the function? He needs only the short Note just mentioned to obtain necessary conditions, which the future will show to be, roughly speaking, sufficient, in order that an integral function be of genus p, in the form of limitations imposed on the moduli of the values of the function for large values of x or also on the moduli of the coefficients C_n of its Maclaurin expansion.

$$f(x) = C_0 + C_1x + \ldots + C_nx^n + \ldots \tag{3A}$$

for great values of n.

The proof of the first limitation is again of a hardly less than childish simplicity. To begin with the simplest case of $p = 0$, that is,

$$f(x) = C\Pi\left(1-\frac{x}{a}\right), \tag{3'}$$

the series $\Sigma\frac{1}{|a|}$ being convergent, it consists in replacing the individual factors

$$1-u=1-\frac{x}{a},$$

or at least a part of them, by the greater quantities $e^{\left|\frac{x}{a}\right|}$. This evidently gives

$$e^{|x|\Sigma\left|\frac{1}{a}\right|}. \tag{3B}$$

But the coefficient of x in the exponent is, on account of the assumed convergence of the series, as small as we like if we leave aside a sufficient number of factors in (3′), and, on the other hand, the product of these factors is for large values of x evidently insignificant with respect to (3B), so that we immediately obtain

$$|f(x)|<e^{\epsilon|x|} \tag{3C}$$

with ϵ arbitrarily small.

This is the base on which the new theory of integral functions has been built.

For $p>0$, the argument is quite similar, the inequality $|1-u|<e^{|u|}$ being replaced by the similar one $|(1-u)e^{Q_p(u)}|<e^{k|u|^p}$, where k is a numerical coefficient.

The valuation of the order of magnitude of the coefficients is a more artificial one. Halphen's works suggest to Poincaré the use of a certain functional transformation, by means of definite integrals, the result of which, if carried out on a holomorphic function limited in magnitude as just found, must be again holomorphic: whence the required inequalities for the coefficients follow. More direct methods have since been easily found for the same deduction.

Poincaré formulates the question whether the necessary

conditions thus found are also sufficient, and he also builds up an example where they prove not to be so. But, really, this example is an exceptional one, the corresponding function being at the limit between two consecutive genera.

I myself have succeeded in proving that, such cases being excepted, the conditions are sufficient, and that, in any case, ambiguity on the value of the required genus can never be of more than one unit. This solves also, but still only *in general,* that is, leaving aside exceptional cases, the two questions also asked by Poincaré, namely:

whether the sum of two integral functions of genus not greater than p is itself of genus not greater than p;

whether the derivative or integral of an integral function having a genus p is itself of genus p.

There still remained the question of the exceptional cases, the question being whether the aforesaid operations could not, in some of them, increase the genus by one, this being the only possibility in contradiction with the general rule. One of the most remarkable facts in Analysis is that this possibility actually exists, as has become known through the beautiful works of Lindelöf, P. Boutroux, and Wiman.

The theory has been beautifully improved from this point of view and others by the researches of Borel and his followers. Numerous new and interesting questions have arisen, such as a more precise study of the minimum value of $|f(x)|$ for a given value of $|x|$, of the relations between the zeros of $f(x)$ and the zeros of its derivatives, the study of the influence of the arguments of the zeros, of the paths along which x may go to infinity and $f(x)$ approach a determinate limiting value; etc. A very careful bibliography of these recent works has been made by Valiron, one of the best workers on the subject, in addition to Borel's new book, *Méthodes et problèmes de théorie des fonctions.*

All this labor, which does not seem to be near its end, takes its exclusive origin in the aforesaid Note of Poincaré, with the addition only of Picard's celebrated theorem.

Improvements in this study of *meromorphic functions,* as brought to light in the beautiful memoirs of Borel and P. Boutroux, come from the same source, for the method used proceeds as in the case of integral functions.

The same should be said for the *essential point,* as treated by Picard and resumed by myself and Maillet.

Then there comes, for uniform functions, the already wide domain of *non-isolated singularities,* one of the most attractive of domains for geometers since Weierstrass. Poincaré gave examples of every known kind of possibility in this respect not by constituting them artificially, but as natural consequences of his researches on the main problems of the Integral Calculus. We have seen that this was the case with fuchsian functions, and that they led him not only to cases of *singular lines,* more general than had been discovered, but to a quite new singular configuration which had never been thought of, namely, *perfect discontinuous sets* of singular points, an intermediary between punctual and linear singularities.

Besides these particular, but especially interesting cases of functions with non-isolated singularities, Poincaré has given a quite general method for the formation of functions with essential singular lines, or, which is equivalent, with lacunary spaces. We have already seen that he most naturally met with such functions while studying singularities of ordinary differential equations and, by the same way, was led to an expression of them. The generalization was obvious and consisted in introducing series of rational functions

$$F(z) = \Sigma \frac{A_n}{z - a_n}, \tag{3D}$$

the poles a_n being distributed on one side of a certain line L and, eventually, along this line. Poincaré, in a later Memoir [1] proved that *in the latter case,* if the series constituted by the numerators A_n is absolutely convergent, every point a_n situated on L is actually a singularity for F. Indeed, if z approaches such an a_n, say a_p, coming from the region R_1 which is devoid of a's, a region inside which F is easily seen to be regular, and along the normal at a_p to L, the unique term corresponding to that point dominates (the word being used as in the existence theorems for differential equations) the sum of all other terms: some, that is all those which correspond to sufficiently great n's, being relatively insignificant because the sum of the A's is smaller than A_p; the others, which are finite in number, because their poles lie further away from z than a_p does.

If, especially, the poles *situated on* L are dense all over this line, L must be singular.

The proof has been resumed and simplified by Goursat.[2] This subject of series of rational functions has been one of the most fruitful in result during these recent years. One of the most unexpected facts which have appeared is that Poincaré's conclusion no longer holds if the poles, instead of lying, in an infinite number, on L itself, are simply dense in its immediate neighborhood. A very simple example given by Wolf [3] shows that a series of the form (3D), with poles distributed inside the unit circle and infinitely dense near its circumference, may represent $1/z$.

But the conclusion again proves to be true (Borel, Den-

[1] Acta Societatis Fennicae, vol. XII, 1883.
[2] Bulletin des Sciences Mathématiques, vol. XI, 1887.
[3] Comptes Rendus de l'Académie des Sciences, November 28 and December 19, 1921.

joy, Carleman) if the convergence of (3D) is sufficiently rapid. Now—a remarkable instance of the connection between various domains in Analysis—such a course of ideas has led Denjoy to a method for solving the problem of *"quasi analytic functions"*—namely, the question of knowing under what limitative conditions imposed on the moduli of the derivatives of a function of the real variable x the analytic continuation is a determinate one: a solution which has become complete since the beautiful improvements brought to it by Carleman, and which the Swedish geometer has recently developed in a course of lectures delivered at Paris.

On the other hand, it is well known that series of the form (3D) have been used by Borel for a generalization of the notion of analytic functions in the complex plane. In the very case considered by Poincaré, namely that of poles dense all over a line, Borel has shown that a kind of continuation, admitting of some of the properties of ordinary analytic functions, may be possible across the polar line. That such a new kind of continuation is nevertheless very essentially different from the ordinary one, has been, however, shown lately by Julia: the latter finds that the branches of a function continued in this way—namely, its values for one and the same value of the variable z—may constitute a *non-enumerable* infinity which is contrary to a well-known theorem, proved by Poincaré himself [1] some years after the period we are dealing with, for ordinary analytic functions.

Thus every part of the theory of uniform functions was improved by Poincaré. But if now we come to the case of *non-uniform* functions, we have not to consider him as

[1] Rendiconti Circolo Matematico di Palermo, vol. II, 1888.

having improved the theory, but as having created it: we have said that it can be spoken of as having started from nothing. The theory now exists, since Poincaré has proved the celebrated *theorem of uniformization:*

"Let y be any analytic function, or, generally speaking, $y_1, y_2, \ldots y_m$ be any—in general, non-uniform—functions of the variable x. There will always exist an auxiliary variable z such that x as well as $y_1, y_2, \ldots y_m$ are uniform analytic functions of z."

The Memoir which contains this fundamental discovery again belongs to the aforesaid vol. XI of the *Bulletin de la Société Mathématique de France,* and resembles the Note "Sur les fonctions entières" by the extraordinary rapidity with which the subject is treated and the decisive result obtained. The tool for this is found in a very simple use of Dirichlet's principle, applied to the Riemann surface S, consisting, in general, of an infinite number of leaves, which represents, in the ordinary fashion, the given functions y, or, more exactly, the analogous surface S', derived from S, not by the ordinary way of dissection, but by a new multiplication of leaves, so as to become simply connected.

This principle, indeed, enables us to construct "Green's function" for any finite portion S_n of such a surface S', namely, a function $u_n(O, P)$ satisfying Laplace's equation, which becomes infinite, and this like $\log 1/r$, only at one inner point O and vanishes at every point of the boundary C_n of S_n. Now, Poincaré will consider progressively such areas S_n', simply connected and more and more extensive, each of which shall include every foregoing one, under which conditions we know that the corresponding values of the function u_n, the two points O, P remaining fixed, will be an increasing function of n. If, moreover, the progressive extension of S_n' is such that any given point P on S'

is finally included within it when n becomes sufficiently great, we can prove that $u_{n'}(O, P)$, where again O and P remain fixed, will approach a determinate limit $u(O, P)$, which will be *Green's function for the whole surface S′;* we can do this if we merely know *any* solution U of Laplace's equation which is infinite, in the aforesaid way, at O and *positive* all over S', such a quantity U being a dominating function for all the u's, and precisely such a dominating function U is afforded to Poincaré by the theory of fuchsian functions.

The function u_n allows, with the help of the corresponding conjugate function v_n, the construction of the quantity $z_n = e^{-(u_n + iv_n)}$, which represents conformally S_n' over the unit-circle; and, similarly, the limiting function u will lead to the construction of a z which will represent S' on the same circle, or, at least,—this is the only thing which Poincaré proved in his original Memoir and is also the only quite essential conclusion—which cannot assume the same value at two different points of S'.

The quantity z, an analytic function of x, will be the required auxiliary variable and the theorem is thus proved.

The idea of such an infinitely complicated Riemann surface may be looked at as an audacious intuition, and this may even be considered as an objection against the full rigor of the argument. But, on returning to the subject in his later Memoir in *Acta Mathematica*, vol. XXXI (1907), Poincaré showed that it could be reduced to the very definition of analytic functions, as given by Weierstrass, by a set of consecutive "elements" respectively defined within mutually overlapping circles. Not only any geometric difficulty concerning the surfaces S, S' and the successive contours C_n was thus eliminated, but such a configuration was precisely the best one for the application

of the "sweeping method," which Poincaré had imagined meanwhile and which gave a new kind of proof for the existence of the successive u_n's.

Meanwhile, Hilbert in his well-known Lectures at the Mathematical Congress at Paris (1900) had insisted upon the importance of Poincaré's discovery, and also upon the improvements which it could usefully receive, especially as concerns the representation of the whole domain of existence of x and y or at least of every regular point in it by inner points of the domain of variation of z whenever regular points are concerned. Unnecessary exceptional points could remain outside the reach of Poincaré's first representation, on account of the choice of the dominating fuchsian function used, as said above, to prove the convergence.

While Poincaré's original Memoir had remained without an echo, and, from 1883 till 1900, nobody had dared to resume the subject as if, even after him, it remained beyond the reach of present science, Hilbert's Lecture was the signal for reattacking it. Not only Hilbert's suggestions were followed, but other interesting questions arose, such as obtaining the simplest representation of the required kind, from which any other one can be deduced by writing $z = f(Z)$, where z is the independent variable corresponding to the simplest representation and Z the other variable. They were treated by Johansson, Brodén and Poincaré himself, while at the same time (1907) Koebe began, on the same subject, a series of researches, the publication of which, in numerous papers, continued for several years. But in spite of the great number of these Memoirs and even of several interesting improvements contained in them, it would, of course, be a mere absurdity to forget as it happened that some German authors did, that all this work

comes exclusively from Poincaré's, and Poincaré's alone, first and decisive impulse.

Further investigations are still of interest, in such a wide subject. Especially, whenever S' is distinct from S, the solution introduces a certain linear group by which the functions $x(z)$ and $y(z)$ are invariant. Such a group, a "fuchsoid group" according to Poincaré's terminology, is analogous to a fuchsian one and likewise discontinuous, with a fundamental invariant circle, but generally derived from an infinite number of generating substitutions. Although Koebe has already devoted some pages to them, the study of such groups could still lead to new results, perhaps especially in concrete cases.

Moreover, as Hilbert pointed out in 1900, all this subject ought to be extended to functions of several variables. But I wonder whether we are able to do so as long as no other Poincaré comes to give us a new measure of the power of the human mind.

We have already said that, generally speaking, in the subject of several variables, our knowledge of non-uniform functions practically amounted to nothing. It only began to exist when Poincaré extended to this case Weierstrass' theorem on meromorphic functions:

"If the function F is such that in the neighborhood of any determinate point it can be expressed by the quotient N/D of two convergent power series, it also admits of expression, which is everywhere valid, as the quotient of two integral functions."

Weierstrass' proof of this theorem for one variable rests on the decomposition of holomorphic functions into primary factors: therefore, it can by no means be extended to the case of two independent variables.

Poincaré's method consists of a kind of analytic *raccordement* between two consecutive representations $F = N_1/D_1$ and $F = N_2/D_2$ of F, respectively valid in two regions R_1, R_2 which are assumed to be contiguous with a common part R' as happens for Weierstrass' "elements." The two given expressions being assumed to be identically equal to each other within R', the question is to find a third one N'/D' valid throughout $R_1 + R_2$ (N' and D' being holomorphic all over $R_1 + R_2$) and coinciding with the two first ones within R'. Poincaré begins by reducing the problem to the research of D' or even of the real quantity $\log |D'|$. For this, he can use the theory of harmonic functions extended to four variables, but not without an apparently serious difficulty, owing to the fact that analytic functions of two complex variables $x = x_1 + ix_2$, $y = y_1 + iy_2$ do not from this point of view behave like analytic functions of one complex variable. The real part of such a function, and therefore the quantity $\log |D|$ must satisfy not only

$$(4) \qquad \nabla^2 u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {y_1}^2} + \frac{\partial^2 u}{\partial {y_2}^2} = 0,$$

but also three other partial differential equations. But Poincaré shows that, after having obtained a solution of (4) connected with the various quantities $\log |D|$ in the required way, it can be brought to satisfy also the three other equations by addition of a holomorphic function, by which the relations in question are not changed.

The *raccordement* between any two consecutive quantities $\log |D|$, in other words the addition of a holomorphic term to each of them so as to let them be the analytic continuation of each other, is effected by means of the solution of Dirichlet's problem in four dimensions and the corresponding properties of potentials of simple and double layers.

The only remaining difficulty lies in the fact that the successive regions are infinite in number, more and more of them generally occurring as the variables grow large indefinitely, so that the corresponding operations, also infinite in number, must be convergent: but this convergency is obtained by Weierstrass' or Mittag-Leffler's classic devices.

This method of Poincaré is perhaps less simple and less direct than those which led him to the results which we have already reviewed, as we see that several intermediaries are required. But this character of simplicity has been fully restored to it in the later work of Cousin.[1] The essential principle remains the same, namely, an analytic *raccordement* between two functions f_1 and f_2 defined in neighboring regions R_1, R_2. But this is effected in the most direct manner on the analytic functions themselves and no longer on their real parts. Let us admit that the two regions correspond to the same values of y, x only being changed—which does not restrict generality, as we can reach every part of the four-dimensional space which represents all the values of x and y successively changing each variable in its turn—f being defined at first as being equal to f_1 in R_1 and to f_2 in R_2, Cousin, in order to make it analytic and continuous, only needs to add the value of the integral

$$\frac{1}{2\pi i}\int_L \frac{f_1(X) - f_2(X)}{X - x}\, dX$$

taken in a proper sense along the boundary L between R_1 and R_2 in the x-plane.

In this way, the question is most elegantly solved, and the solution extends itself to any number of variables,

[1] Acta Mathematica, vol. XIX.

which could not have been the case for Poincaré's original method without a revision of several details.

Let us not leave the Memoir of Poincaré without noticing that he also suggests at the end of the Memoir a question which could be attacked by similar methods: y being any non-uniform function of x, subject only to the assumption that it possess no essential point at a finite distance, the method used in the Memoir could be used to find a holomorphic function $G(x, y)$ such that

$$G(x, y) = 0$$

precisely defines the given function $y(x)$. This question and the neighboring subject of elimination between transcendental equations would certainly require further work. The questions are certainly not quite simple, as Osgood shows by the example of the three equations

$$x = u, \qquad y = uv, \qquad z = uve^{v},$$

between which no elimination of u and v can be carried out, at least by a final equation $G(x, y, z) = 0$ such that $G(x, y, z)$ be holomorphic in the neighborhood of the origin. This, however, would not contradict Poincaré's statement as we have here the case of an essential point.

Besides these results in the general study of analytic functions, Poincaré, during the same period, devoted one Memoir—it was the third in *Bulletin de la Société Mathématique de France,* vol. XI—to a study on abelian functions, more exactly, theta functions: a study of an altogether novel character. The question concerns the system obtained by equating simultaneously to zero n given theta functions, with the same periods, in n variables, and consists in finding the number of distinct solutions, that is,

solutions not differing from each other by periods. Poincaré finds this number by a method quite intuitive in character. This is also the first time in his work where he applies Kronecker's theorem for the number of solutions of a system of equations inside a given domain, an admirable instrument the power of which he immediately recognized, and which enabled him to obtain many other victories.

The result thus established was for him the starting point of a series of researches of an entirely new kind, the origin of which lies in the fact that theta functions in p variables, as depending on $p(p+1)/2$ parameters, are more general than theta functions giving the inversion of abelian integrals along an arbitrary algebraic curve of genus p, which class only includes $3p-3$ parameters. The latter theta functions are, therefore, *special:* to characterize these special theta functions amongst general ones, is the new point of view considered by Poincaré and which led him to striking results concerning abelian functions and algebraic varieties.

Cases of reduction—a part of the theory of abelian functions which was being most attentively studied on account of certain remarkable statements of Weierstrass which had just become known to geometers and which Poincaré succeeded in proving—are connected with the above remark: for, with a theta function in p variables, depending on $p(p+1)/2$ parameters, we can construct an algebraic curve, the genus of which will be generally greater than p, as the most general class of curves of genus p depends on a smaller number of parameters, though, by its very construction, the corresponding theta functions can be expressed in p variables.

Let us recall that another improvement was brought by Poincaré to our knowledge of functions of several variables a few years later, namely, the extension of Cauchy's notion of residue. Now, these two works of Poincaré followed by the above mentioned thesis of Cousin have remained for years the only attempts, since the Vorbereitungssatz, of Weierstrass, to penetrate into that part of the theory of analytic functions. Only by quite recent works, such as those of Faber, Hahn, Hartogs, E. E. Levi, have new ways been opened and new valuable results obtained in that line.

DIFFERENTIAL EQUATIONS AND CELESTIAL MECHANICS

THE integration of differential equations, whether ordinary or partial, is the central problem of modern Analysis and remains a highly difficult one. This is above all a case where Poincaré's word is valid: "Il n'y a pas des problèmes résolus et d'autres qui ne le sont pas; il y a des problèmes plus ou moins résolus." Leaving aside exceptional equations the integration of which occurs in elementary fashion, no general method enables us to master fully the solution; and we can only try to get partial aspects of it, such as the following ones, in which attention is confined to ordinary differential equations.

I. A first "general" method, the outline of which had been pointed out by the geometers of the eighteenth century, was scientifically built up by Cauchy. Limiting ourselves, for the present, to analytic equations, we know, at least theoretically, how to construct power series which represent the solution locally, that is, in the neighborhood of any regular place. Of course it is known that this does not permit us, otherwise than very indirectly, to represent the solution within its whole domain of existence.

After having been thus established for the perfectly regular case, Cauchy's result had to be extended, under proper conditions and proper modifications, to singular places: this was the work of Briot and Bouquet, of Fuchs, and others.

II. Of course, cases of integration through elementary

functions, when not classic, although most unlikely to be met with in any determinate problem, may still be looked for.

III. The problem being out of the reach of elementary functions, we can see whether we cannot reach the solution by properly defined and studied categories of new transcendentals.

IV. A direct study of the unknown functions from the point of view of the general theory of analytic functions is important as well in itself as for the very application of the method just mentioned; for example, for the choice of the required transcendentals.

V. If we cannot express the solution in finite terms, the question arises whether other series than Cauchy's power series could not have a wider reach and be valid for all values of the variable.

We shall see that these trails are not the only ones which can be imagined; but they were practically the only ones which were known before Poincaré, and even the last of them had not been studied from a general point of view.

Poincaré's genius could not but be immediately attracted by the great problem of differential equations. All the above-mentioned methods, all methods investigated by his predecessors, were followed by him, and this already in the early period we are dealing with, while, at the same time, new and unimagined ways were opened by him.

To the "local" study of solutions, we have seen that he devoted himself at first, in his thesis, and we have also seen that the improvement thus obtained is a profound and essential one.

Although the research of cases of elementary integration is, in the present state of science, a secondary thing,

he gave, but later on, new principles for the discovery of algebraic solutions; and, on the other hand, no geometer may evidently be said to have created a more powerful instrument for the expression of solutions of algebraic differential equations than he who constructed the fuchsian functions.

We shall now see how Poincaré began his contribution to our knowledge of the functional properties of solutions of differential equations, and, with this aspect of the problem, we shall perceive one of the great principles with the help of which all its other aspects were transformed by him.

Fuchs had considered, for the first time, the simplest hypothesis which may be imagined as to the, generally unknown, distribution of the singular points of the solutions of a differential equation, namely, the case of an equation of the first order whose singular points do not vary with the constant of integration; and he had succeeded in writing, for an algebraic equation, a system of necessary and sufficient conditions for this. But this was only a partial answer: for there remained the question of whether such equations actually exist and lead to new transcendentals. It was this answer which was given by Poincaré, and it proved to be a negative one, every equation of the class considered by Fuchs being reducible to known types.

Now such a negative result happens to be anything but sterile and opened to science a long way, the end of which we are even yet far from seeing, and this course we shall now try to explain.

It might be said that the difficulty of a mathematical problem lies in relation with the number of its solutions, with its "ambiguity" as Galois would have said. The case

of ordinary algebraic equations most distinctly shows us that such is the intimate reason of the difficulty, the required formulæ being bound to be such that they can lead to any one of the, algebraically inseparable, unknown roots. If we take an ordinary differential equation, the number of the solutions is infinite; and no doubt the circumstance is connected with the difficulty of discovering them, which difficulty increases with the order of the equation; that is, with the number of arbitrary constants introduced by the integration; and a similar remark evidently applies to partial differential equations.

Now, one who reflects on the history of science will see that its progress has been much delayed by not clearly perceiving this true reason of the difficulty and not attacking it directly. Formulæ for solving the equations of the lowest degrees could be found by reasoning on *one* root; but no gain in the general theory could be obtained in that way. After Abel and Galois, we may say that we know everything about the algebraic solution of equations but we should not do so if they had not taught us to introduce simultaneously *all* the roots and to deal with their mutual relations.

The history of the theory of differential equations is quite similar to the preceding one. In one case as in the other, geometers only thought of one solution. It is only since Poincaré that we have come to know how insufficient is such a point of view as this.

In the case of Fuchs' above-mentioned problem, Poincaré will not even let the independent variable x vary: he will consider two values x_0 and x of this variable. The corresponding values of the unknown function being denoted by y_0 and y, the latter will be considered not as a function of x, but as a function of y_0; and, by means of

the given hypothesis, it can be shown that the correspondence between y_0 and y is a birational one: whence the conclusion easily follows.

We now see what a new method of investigation was thus introduced in the theory, and we can conceive how this Memoir of Poincaré was capable of generating such extensive and fruitful researches as Painlevé's. These, as is well known, have been pursued by many geometers of the present generation, such as P. Boutroux, Gambier, Garnier, in France, Richard Fuchs in Germany, and others, who have let us know more and more accurately the properties of Painlevé's transcendentals; and it seems to me that the question may perhaps be still "more" solved than it is now at least as concerns an effective calculation of the unknown. Should we be satisfied, in the case of Weierstrass' theta function, for example, with such construction as we have of it in its Maclaurin expansion?

The way which consists in replacing Cauchy's series by more generally convergent ones has also been opened by Poincaré. In one of his earliest Notes,[1] considering the differential system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}, \tag{5}$$

he observes that if we define an auxiliary variable s by equating the common value of the above ratios to

$$\frac{ds}{1 + X_1^2 + \ldots + X_n^2}, \tag{5A}$$

the equations will behave regularly for every real value of s, and that even the x's will be holomorphic functions of s throughout a certain strip of the s-plane on both sides of

[1] Comptes Rendus de l'Académie des Sciences, Paris, February 27, 1882.

the real axis: this being acquired, if we introduce the new variable

$$u = \frac{e^{\alpha s} - 1}{e^{\alpha s} + 1}, \tag{5B}$$

where α is a suitably chosen positive constant, and consider the x's as functions of u, they can be expressed by Maclaurin expansions in powers of u, convergent within the circle which corresponds to the above-mentioned strip and therefore reaching every real value of s.

It was the first time that such an attempt had been made, that other series than Cauchy's were proposed for other than special kinds of problems. It was a natural thing to apply such a principle to the celebrated problem of three bodies. In this case, it can be shown that the method leads to expansions which are valid for every value of the time, provided it never happens that two of the bodies come into collision.

This, of course, was a formidable gap in the face of which the result seems to lose much of its value: for the existence or non-existence of a collision between two bodies at a later time of the motion is a thing which we are not able to foresee mathematically from the initial conditions. Mathematicians know how the researches of Painlevé, and later of Levi-Civita, Bisconcini, Armellini culminated in Sundman's Memoir. Sundman shows that when two and only two of our material points meet, say at a certain time t_0, the functions which represent the coördinates in terms of t admit of an analytical continuation after $t = t_0$,—an operation the first instance of which is again found in Poincaré's work,[1]—on properly choosing a new independent variable; and, by application of Poincaré's above-mentioned

[1] "Les Méthodes nouvelles de la Mécanique céleste," vol. III, Chap. XXVI, p. 168.

ideas, Sundman is able to discover a variable in powers of which t itself and the coördinates can be expanded in convergent series, the series for t assuming every positive value from 0 to infinity.

All this, however, depends on the hypothesis that no simultaneous meeting of all three bodies can occur. But this hypothesis does not give place to the same difficulty as the former: we can actually see whether it is satisfied or not; or, at least, there is a sufficient condition for this in terms of the initial data, namely, that not all three constants of areas vanish.

This solution of Sundman is one of the most capital improvements in theoretical Celestial Mechanics for the last half century. But it is not the only case where the above-mentioned method of Poincaré has been useful in the study of the problem of three bodies. Let us only mention that one of our younger geometers, M. Chazy, has successfully used a variable quite analogous to the quantity u defined by (5B). Moreover, several other principles due to Poincaré have been applied in Chazy's work, which may be reckoned among their most beautiful applications.

On the other hand, the problem of three bodies, in spite of its great difficulty, may be considered as a special one in differential equations and, independently of the quite general methods such as the above, can be treated by those which are classic in Celestial Mechanics. The most important of these consists, as is well known, in the expression of the required coördinates by trigonometric series.

The series in question are thoroughly different from Fourier's, though having with them some properties in common: the multiples of the time occurring as arguments under trigonometric signs are not proportional to integral

coefficients, but, on the contrary, such coefficients of t may as well decrease as increase indefinitely. No systematic study of such series had been made before Poincaré and, nevertheless, astronomers proposed to use them (for instance, those of Lindstedt) for the proof of such important conclusions as the stability of our solar system. It seemed an evident thing, and was indeed universally admitted as such, that if Lindstedt's series, the terms of which contained only sines and cosines, could be proved to be convergent for every t (which, however, was and remains more than doubtful), this would be sufficient to prove the required stability.

Poincaré was the first to remark that this conclusion was by no means a certain one, and, moreover, that nothing could be asserted before having undertaken a general and mathematical study of the aforesaid kind of series

$$(6) \qquad \Sigma\,(C_n \cos \alpha_n t + C_n' \sin \alpha_n t),$$

where the constant coefficients α, which are infinite in number, may have any values, indefinitely small as well as indefinitely great ones. The C's are also constants.

Let us admit that such a series (6) is always convergent. If this convergence is *uniform,* at least for all positive t's, the function thus represented remains finite for $t = \infty$. But such *uniform* convergence is by no means certain, as Poincaré finds in his paper of the *Bulletin Astronomique,* vol. I (1884), again a work remarkable for its shortness and simplicity in comparison with its fundamental importance.

Before investigating the question of uniformity, he first considers the question of absolute convergence, and up to that point the conclusion is peculiarly simple: if the series

(6) converges absolutely throughout a certain interval, however small, it does so for every real t.

But the two kinds of terms in (6) nevertheless behave quite differently. The cosine series $\Sigma C_n \cos \alpha_n t$ cannot converge absolutely for $t = 0$ without converging uniformly, and this is not at all the case for the series whose terms contain sines, as is immediately shown by such examples as

$$\Sigma\, 2^n \sin\left(\frac{t}{3^n}\right), \tag{6A}$$

which series always converges, the asymptotic value of the general term being $\left(\frac{2}{3}\right)^n t$, though any term, however great n be assumed, can be made equal to 2^n by suitably choosing t. Generally speaking, if the α's are greater than a fixed number, the condition for absolute convergence is that $\Sigma|C_n|$ converge, from which uniform convergence also follows; but if the α's are smaller than a fixed number, and generally liable to approach 0 indefinitely, the condition for an absolute convergence is the convergence of $\Sigma|C_n\alpha_n|$.

In this case, as we have just seen, uniform convergence is by no means evident; and nothing proves that the value of the series remains finite, for indefinitely increasing t's. But can even the contrary conclusion be proved? This is what Poincaré does with the help of formulæ analogous to Fourier's. Any coefficient C in the trigonometric expansion of $f(t)$ is expressed with the aid of a mean value of $f(t) \begin{matrix}\cos\\ \sin\end{matrix} (\alpha_n t)$; only this mean value, instead of being taken over a period (which does not exist, in general, in the present problem), is to be taken over an indefinitely increasing interval, namely,

$$\lim_{T=\infty} \frac{1}{T} \int_{-T}^{+T} f(t) \begin{matrix}\cos\\ \sin\end{matrix} (\alpha_n t)\, dt.$$

This formula shows that $f(t)$ cannot have its absolute value everywhere less than $C_{\frac{1}{2}n}$ or $C_{\frac{1}{2}n'}$; and if there are indefinitely increasing coefficients in (6), as happens in (6A), the value of $f(t)$ *cannot* remain finite.

Thus, these few pages in the *Bulletin Astronomique* have upset all former conceptions of astronomers in the problem by showing that the convergence of trigonometric series such as Lindstedt's, even proved for every t, is by no means a sufficient condition for stability.

However, the mere fact of a systematic study of the general series (6) has not remained fruitless for Celestial Mechanics. In connection with other suggestions given by Poincaré, in later volumes of the *Bulletin Astronomique* it has generated the theory of quasi-periodic functions, such as developed by Bohl and Esclangon, which seems to be one of the most powerful instruments for the improvement of this part of science, especially if further steps can be taken in the solution of the difficult and curious analytical questions which it raises.

These consequences of the first astronomical Note of Poincaré are the more important to notice because they concern the positive point of view, while the immediate conclusions of the Note are, as we have just seen, of a negative tendency, that is, they chiefly show us that some means imagined for attacking the problem do not deserve the first confidence we had in them. Such negative conclusions, occurring to Poincaré several times in the work of the following years, have enabled us to see how remote the solution of the problem is. But from this very time and in the same volume of the *Bulletin Astronomique,* a result of a positive nature also appears, and even introduces one of the most important means by which, since

Poincaré, we may hope to penetrate into that solution, we mean *periodic motions*.

The American astronomer Hill had obtained[1] the first kind of cases, classic ones being abstracted, where the motion of a "moon," that is, of a body with a very small mass gravitating under the combined influence of two greater masses which revolve about each other, a motion depending on the so-called restricted problem of three bodies, proves to be periodic. But it is Poincaré who not only found extensive classes of such motions, but also discovered the general importance of them, by an application of the important principle which we have pointed out above, that is, by investigating the relations between the motions thus obtained and the other ones which also satisfy the differential equations, so that the former, when once known, are used for the study of the latter; indeed, he has been justified in writing that such periodic motions have proved to be "the only breach through which we have been able to approach a fortress hitherto considered as inaccessible." In several circumstances, and especially in some of my own researches inspired by Poincaré's methods, they have behaved as a kind of system of coördinates to which other trajectories could be referred.

This first paper on the subject is devoted by Poincaré to the construction of a simple class of periodic trajectories. He again applies the remarkable theorem of Kronecker on systems of equations already alluded to. As a matter of fact, however, this theorem is not necessary for the conclusions stated in that case: they all derived from the classic theorem on the existence of the solution for a system of n ordinary equations in n unknowns, the Jacobian of which does not vanish. Poincaré's argument may even be said to

[1] American Journal of Mathematics, vol. I, 1878.

imply a new proof of that theorem, which indeed would be of only mediocre interest if Poincaré had not met with a curious intermediate proposition by which the existence of the required solutions can be asserted inside a certain cube whenever we know that each of the left-hand members of the given equations has contrary signs on two of its opposite faces.

But finally, for the astronomical application, the classic property of the Jacobian is the only one that is needed. Let us see how it is used.

The starting point is a well-known one in Celestial Mechanics, consisting of the remark that the given differential equations depend on a parameter μ, namely, a disturbing mass,[1] the value of which will be very small, the integration, on the other hand, being elementary when we replace this parameter μ by zero. Astronomers generally succeed in calculating expressions which satisfy the equations up to a certain order in μ, thus giving an approximate solution of the system. In the present case, an exact solution will be found, or at least its existence proved, for every value of μ sufficiently small.

Poincaré assumes that, for $t = 0$, the three bodies start from what he calls a "symmetrical conjunction." This means that, initially, the three velocities are normal to the plane of the three bodies, if motion takes place in three-dimensional space; and, if the motion is a plane one, that, initially, the three points are collinear and their velocities perpendicular to the straight line which contains them. It follows therefrom that the motion must be symmetrical with respect to the plane (in the former case) or the straight line (in the latter) in question, so that cartesian

[1] More exactly, Poincaré assumes two of the masses to be very small and proportional to μ.

coördinates perpendicular to this plane or this straight line will be odd functions, and coördinates parallel to them, even functions of t. Let us now assume that, for another value of the time, say $t = \pi$, which does not restrict generality if the unit of time is properly chosen, a new symmetrical conjunction takes place, on a new plane or a new straight line which cuts the original one at a certain angle α. We see that the motion will admit of two symmetries, therefore, of a rotation of angle 2α, corresponding to the change of t to $t + 2\pi$: we shall have a periodic motion, at least if we refer it to properly rotating axes.

Whether this takes place or not, depends on the initial data: the requisite conditions for this, or generally, the conditions for a periodic motion, will be expressed, as Poincaré finds, by several equations

$$X_i(x) + 0 \qquad (i = 1, 2, \ldots), \tag{7}$$

x generally meaning the parameters which define the initial data, which we should be able to write if we could integrate our differential equations. In fact, we cannot, for an arbitrary μ; but we can for $\mu = 0$.

Now, let us arrange matters, by limiting ourselves to special cases if necessary, so that:

(a) there are exactly as many equations (7) as unknowns;

(b) their Jacobian, *for the value zero of the parameter,* does not vanish.

The fundamental theorem of implicit functions proves that there will be a unique solution for every sufficiently small μ.

Thus, to apply such a method, we do not need to know anything but the nature of the equations (7) for $\mu = 0$;

and this is actually known, as corresponding to the ordinary Keplerian motion.

In the Note we are speaking of, Poincaré applies this principle to the double symmetrical conjunction above considered, in three kinds of cases:

The first kind will correspond to zero inclinations and very small eccentricities;

the second one, to zero inclinations and finite eccentricities;

the third one, to finite inclinations and very small eccentricities.

The method appears very simply in the first case. The motion is then considered exclusively in the xy-plane. We assume, as understood by the above, that two of the masses contain μ as a factor: therefore, for $\mu = 0$, they are sensibly zero and no mutual action exists between them, so that they both revolve around the third mass, according to Kepler's laws. Now, let us assume initially, that is, for $\mu = 0$,

$$e = e' = 0,$$

e and e' being the eccentricities, so that the two motions are circular; and

$$n - n' = 1,$$

n and n' denoting the mean motions,[1] but, on the other hand, that n and n' are not integers. Then, it is clear that there will be a second symmetrical conjunction for $t = \pi$, which can be expressed by saying that the above values of e, e', n (the other mean motion n' being considered as given beforehand) satisfy

$$X = Y = Z = 0,$$

[1] It is also understood that the radii of the two circular orbits must be connected with the motions by Kepler's third law.

where X, Y mean the derivatives of the two radii vectores with respect to t for $t = \pi$, and Z the difference of longitude for the same value of the time. But, on the other hand, we can assert that the Jacobian of X, Y, Z with respect to e, e', n is not zero, under the aforesaid conditions; for an elliptic radius vector, if not axial, which is excluded by the hypothesis that n is not an integer, admits of a derivative X or Y which contains e (or e') as a factor in the first power only and, for $e = e' = 0$, the difference of longitude Z is proportional to $(n - n' - 1)$. Thus, the above principle shows that a periodic motion of the kind considered exists for every sufficiently small μ.

Poincaré calculates the terms of the first order in μ in the trigonometric series in t which express the motion.[1]

In the two other cases, the question is less simple. We cannot assume the functional determinant of the left-hand members of the equations which express the double symmetrical conjunction for $\mu = 0$ to be different from zero: on the contrary, one of these left-hand members, say X, is identically zero with μ. In such cases, X containing μ as a factor, the method will consist in replacing X by $X/\mu = X_1$ and calculating the value of X_1 for $\mu = 0$, that is, the coefficient of the first power of μ in X: which, if not so simply done as in the preceding case, can nevertheless be done by quadratures,[2] where, of course, account must be taken of the variation of the Keplerian elements, such as defined by the classic differential equations of Celestial Mechanics, but reduced to its infinitesimal part of the first order.

[1] Poincaré notices that the calculation fails for integer n's: we have seen, in the text, that such is also the case for the very existence of the solution or, at least, its proof.

[2] When resuming the subject, some years later, in his "Méthodes nouvelles de la Mécanique Céleste," vol. I, Chap. III, Poincaré obtains the same result more simply by reducing it to considerations of minimum.

Such is the first method imagined by Poincaré for the construction of periodic solutions for the main problem of Celestial Mechanics. Other ones, of most different kinds, did not cease to appear over and over again in his later work for the study of this question which, after him, is definitely considered as one of the most capital ones in the theory. His first contact with it was, as we see, almost contemporaneous with his first writings; and the wing of death was hovering around him when he dealt with it for the last time, in a Memoir the publication of which [1] he did not see, and which is brought to completion, as we know, by Birkhoff's universally known Memoir.

This importance of periodic solutions in Poincaré's mind and, consequently, in present science, is connected with the new point of view which now remains for us to examine.

[1] Rendiconti Circolo Matematico di Palermo, vol. XXXIII, 1912.

THE QUALITATIVE THEORY

THIS new point of view for the study of differential equations had hardly been thought of before Poincaré. It consists in remembering that mathematics is not only the science of magnitude, but also of order.

In elementary problems, such as those considered in the first ages of science, these two parts of the question are treated simultaneously, and even the former generally includes the latter, every detail of information on the qualitative properties of the solution being easily deduced from the formulæ which express it quantitatively.

But matters already appear in quite an opposite light when attacking the first problem which has been characteristic of modern mathematical science, we mean the solution of ordinary equations, transcendental or algebraic, other than linear or quadratic. Here, let us quote Poincaré himself:[1]

"Pour étudier une équation algébrique, on commence par rechercher, à l'aide du théorème de Sturm, quel est le nombre des racines réelles, c'est la partie qualitative; puis on calcule la valeur numérique de ces racines, ce qui constitue l'étude quantitative de l'équation. De même, pour étudier une courbe algébrique, on commence par *construire* cette courbe, comme on dit dans les cours de Mathématiques spéciales, c'est-à-dire qu'on cherche quelles sont les branches de courbes fermées, les branches infinies, etc. Après cette

[1] Journal de Mathématiques pures et appliquées, 3rd series, vol. VII, 1881, p. 376.

étude qualitative de la courbe, on peut en déterminer exactement un certain nombre de points."

How such a distinction may be easily overlooked, is often clearly seen in teaching. As a matter of fact, the phrase: "To *trace* a curve"—"construire une courbe"—used by Poincaré in the above lines, is most often curiously misunderstood, it being forgotten that it can admit of two quite different meanings; errors arising from this fact may even be committed by advanced students, as I have personally had the opportunity of experiencing.

At any rate, the problem of algebraic equations gives us a first and an evident instance of the fact that the order between the two (qualitative and quantitative) parts of the study, such as was usual to geometers in elementary questions, is to be reversed when taking higher subjects: in this case, already, qualitative study must go first, and, so to speak, independently of the quantitative calculation.

Similarly, as Poincaré was the first to see clearly, a fundamental and necessary problem in Integral Calculus is: to trace, in the qualitative meaning of the word, the curves defined by differential equations. Not only must such a qualitative tracing precede the construction of series or quantitative expressions and be a guide in their choice; but it is also essential in itself, as several most important questions in Analysis and Mechanics are of a qualitative nature.

Such is, above all, the celebrated question of the stability of the solar system so eagerly investigated ever since the creation of Celestial Mechanics. One may wonder at the fact that the existence of such a capital problem had not turned minds towards the research of general and systematic methods in that line: which is the more astonishing because some remarkable and fruitful attempts of that kind have existed. Thus, the proof, by Dirichlet, of Lagrange's

theorem of stability of equilibrium is evidently a qualitative study. Then we may recall Sturm's now celebrated theorems and methods on linear equations, and finally, Liouville's theorem on what is called—since Poincaré—"integral invariants." But even this enumeration shows by itself how little such examples had been followed, in spite of the beauty of the results obtained; and even the last of them has been so completely overlooked for years that the theorem had to be rediscovered several times (by Boltzmann, Gibbs and Poincaré) before being known to geometers as a capital result in Analysis, Mechanics and Physics.

The main reason for that is certainly that the attention of geometers, for a quarter of a century, was entirely attracted to the beautiful theory of analytic functions, the unexpected and splendid results of which seemed to cover the whole domain of mathematical science. It had revealed to geometers that "the shortest way between two truths in the real domain very often goes through the complex one."

This, indeed, is most frequently true for quantitative results, but is rarely so in qualitative questions. There, the marvellous instrument created by Cauchy generally fails. Poincaré, who had used it so powerfully in his preceding researches, could not reckon on its help in the new work he was undertaking.

The new means which he had to create in order to replace it chiefly consist in the principle which we have mentioned above, and which was already used in Sturm's and Liouville's works; the collective, and no longer individual, introduction of the different solutions of the equation, with the consideration of their mutual relations.

Of course, the first study will concern one differential equation of the first order

$$\frac{dx}{X} = \frac{dy}{Y} \tag{8}$$

where, as usual, we shall admit the common value of the two ratios to be

$$= dt. \tag{8'}$$

Poincaré will begin by considering an equation which is not only of the first order, but also of the first degree, so that dy/dx will be expressed by a rational—and, consequently (the only thing which is important) a one-valued—function of x, y. Then, with an arbitrary point of the plane—or, what is equivalent and what Poincaré prefers, in order to deal better with infinite values of x, y, with a point of the sphere into which the plane can be projected [1] —there is associated a perfectly determinate direction (X, Y), namely, the direction of the tangent to a curve of the required kind. Poincaré will proceed (as a general method) *by letting t increase;* a procedure which is justified by the fact that exactly one of the required curves goes through that arbitrary point, and thus yields a direction of increasing t's.

Now, we may say that the only principle which will enable Poincaré to solve the problem is the property just mentioned, or, which is equivalent, the following one:

(a) Generally, that is, at an arbitrary point of the sphere, no two curves satisfying the equation can intersect, a form of statement showing clearly that a relation between various solutions is concerned. More strictly speaking, he needs only to combine this property with the principle:

[1] The projection is made gnomonically, that is, with the centre of the sphere as centre of perspective, rather than stereographically, which would introduce a singularity corresponding to the line of infinity in the plane.

(b) at any point of an arbitrarily given line L, not a solution of the equation, we can say in what direction the corresponding curve satisfying (8) will cross L.

These two starting points alone are sufficient for Poincaré's analysis. In spite of their simplicity, they will lead to results which will show how difficult the problem was in ways no geometer could have foreseen.

Geometers, of course, were inclined to think that an idea of the general equation (8) could be got by our knowledge of the special equations which we are able to integrate completely. Now, this point of view proves to be utterly misleading. In spite of every analogy between two equations of the type (8), one of which can be integrated and the other cannot, this very difference ought to let us presume that they must have different properties, and Poincaré's results will show that the differences are generally quite essential ones. This will even appear from the very first steps of the theory.

The beginning of it consists in noticing that (a) fails at some special points, the so-called "singular points" of the equation, where the direction (X, Y) becomes indeterminate by simultaneous vanishing of X, Y (for points not on the "equator" of the sphere, corresponding to points at a finite distance on the plane, the case of points at infinity being, as usual, reconducted to the former by a suitable homographic transformation).

Such singular points were well known from Briot and Bouquet's researches; but here, a new discussion is necessary, as Briot and Bouquet did not insist on the distinction between real and imaginary, and we must. From this new point of view, singular points (exceptional cases, which correspond to higher singularities being left aside) belong to four possible kinds:

1° *nodes,* each of which is the common limiting point (for $t = +\infty$) of an infinity of solving curves of (8):— indeed, of all solutions coming sufficiently near it (as happens for the origin of coördinates when $X = -x$, $Y = -y$ or, more generally, when $X = -sx$, $Y = -s'y$, with s and s' being positive)[1];

2° *passes,*[2] around which our curves will behave qualitatively like hyperbolas $xy = \text{const.}$ (example: the origin of coördinates, for $X = sx$, $Y = s'y$, when s and s' have contrary signs);

3° *foci,* which the required curves approach, as they did for nodes, but now *spirally,* that is, by turning around them in the manner of a logarithmic spiral (example: the origin, when $X = -x - ky$, $Y = -y + kx$);

4° *centres,* each of which is surrounded by closed curves all satisfying (8) and including each other in the way of homothetic ellipses (example: $X = -y$, $Y = x$).

These four possible dispositions are obtained by a *direct* study of the singular point, the use of (b) already proving useful. Now, which of them generally occur in the cases where elementary integration is possible?

Let

$$F(x, y) = \text{const.} \tag{8A}$$

be the general integral of the equation. If we introduce a third coördinate z, which may be considered as an altitude ($z = 0$ being taken as the horizontal plane), we may speak of the lines (8A) as the *level lines* of the topographic surface $z = F(x, y)$. Now, the singular points of a system of ordinary level lines are well known. They are given

[1] If s and s' were negative, the origin would practically behave like an ordinary point, if we still agree to follow trajectories only in the direction of increasing t's.

[2] "Cols" is the French word used by Poincaré.

by the summits or depressions, such as the bottom of a lake, of the topographic surface: the two former classes corresponding to maxima or minima of F evidently give *centres* in the present terminology, while our "passes"—minimaxes of F—are nothing else, in this case, than the points which the topographer denotes by the same name.

Of course, it is possible to find forms of F such that nodes or foci occur: we have only to take

$$F = \frac{y}{x} \quad \text{or} \quad F = L\sqrt{x^2 + y^2} - K \arctan \frac{y}{x}$$

(as would be obtained by integrating

$$\frac{dx}{-x} = \frac{dy}{-y} \quad \text{or} \quad \frac{dx}{-x - ky} = \frac{dy}{-y + kx}\Big).$$

In these cases, F appears under the form 0/0 at the singular point.

If, now, instead of level lines of an ordinary topographic surface, we consider *slope lines,* which are precisely defined by a differential equation of the type (8), it is well known that the usual dispositions are different: no centres exist; we should find *nodes,* corresponding to depressions (and also to summits, if we should follow our lines in the ascending direction), and then *passes.* Moreover, a greater complication than known in usual integrable cases already appears in the fact that an arbitrary slope line generally ends at a node (or in other words at a depression), but not always at the same node: there are several minima of F, and the slope line will approach one or the other of these points, according to the choice of its starting point within one or the other of corresponding regions into which the topographic surface is divided. The construction of an integrable differential equation leading to such dispositions would already be a problem of some complication.

This would show us, so far, nodes, passes and centres as the most usual singularities, liable to occur for an arbitrary given differential equation.

Now, Poincaré's direct discussion, carried out on inspection of the equation itself without inquiring whether it is integrable or not, leads to a quite different conclusion. The fact is that usual singular points belong to the first three above mentioned classes: namely, nodes, passes or foci. On the contrary, centres are met with only exceptionally: their existence is subordinate to a series, even an infinite series, of necessary conditions. A centre appears as a

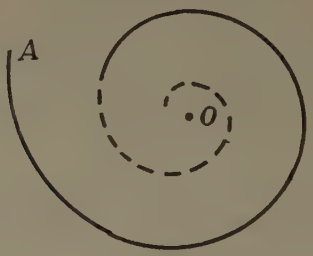

Fig. 1

quite exceptional subcase of a focus. If a trajectory, that is, a curve satisfying our equation [1] starts from any determinate point *A* in the neighborhood of the singular point O and turns around O, it *may,* after one complete turn, go again through *A,* which will give a closed curve. But generally, it will not do so: and the less if we require it to do so for every position of *A* in the vicinity of O; so that generally, the new arc of our trajectory will lie either *inside* or *outside* the original one; and if, for instance, we have the former case (fig. 1), the third arc, after a second complete revolution around O, will again be inside the second

[1] Poincaré uses the word "caractéristique" or, more precisely, "demi caractéristique," when taking account, as we do in the text, of the fact that the curve is to be followed in one direction only.

one, otherwise it would be bound to cross it, which would contradict (a), and so on. It is clear that this will give a focus provided these successive arcs indefinitely approach O—the contrary case remaining to be examined presently, as depending on the general behavior of trajectories when no longer in the vicinity of a singular point.

Before coming to this main part of the problem, the question arises whether singular points like those of which we have just spoken, necessarily exist. To this, the answer is an affirmative one. Poincaré obtains it, in the first place, by direct algebraic computations; but, later on, he is enabled to reason independently of the polynomial nature of X, Y, by introducing a kind of consideration which is fundamental to the subject, I mean considerations of Analysis Situs. For the present object, the conception which he uses is the one of an *index* closely connected with Kronecker's and defined, practically, by the total rotation of the vector (X, Y) when the corresponding point describes a closed line L. As happens in Kronecker's case, such an index gives precise information, leading to the above conclusion, concerning the singular points which may or may not lie inside or outside L. It finally proves impossible that on a surface of genus zero, such as the sphere is, any distribution of vectors such as (X, Y) could be devoid of singular points.

We immediately add that these questions of Analysis Situs play a more and more important part as Poincaré will proceed further in these researches, or as other geometers will continue his work. Especially, the above hypothesis that our differential equation is of the first degree, in other words, as we have seen, that it corresponds to a vectorial distribution over a sphere—is by no means *the* necessary one: the essential in it consists of the fact that

the surface, over which the distribution is given (and this in a univocal way, so that there is but one direction associated with an arbitrary point), is a simply connected one. For any equation—whether of the first degree or not—satisfying this geometric condition, the theory which we are now speaking of is valid.

This intervention of Analysis Situs is particularly opportune in the second part of the theory, where the trajectories are examined elsewhere than in the neighborhood of singular points, and here the results of the study prove to be altogether unanticipated. The disposition which such study brings in evidence differs still more than the previous ones from what could be expected by known examples of integrable cases. This disposition is as follows:

There are several closed curves, or, as Poincaré says, "cycles," satisfying (8), on the surface of our sphere.

Any other trajectory—if not ending at a singular point, or, what amounts to the same thing, approaching it indefinitely, in the case of a focus—approaches spirally such a closed trajectory, that is, indefinitely turns around it at the same time as it comes nearer and nearer to it, as illustrated in fig. 2; and accordingly these closed trajectories are called by Poincaré "limiting cycles." [1]

The proof of this remarkable fact becomes quite intuitive when an algebraic curve L has been constructed which cuts the trajectory in question C at an infinite number of points: and Poincaré begins by showing that such an algebraic curve must exist, if C is not closed.[2]

[1] Some closed trajectories are limiting cycles for increasing t's and some others for decreasing t's.

[2] Such a curve can be very simply deduced from the consideration of any regular function Φ (x, y), with the help of principles used later on by Liapounoff (Kharkow, 1892; translated into French in the Annales de la Faculté des Sciences de Toulouse, 2nd series, vol. IX) and by ourselves (Journal de Mathématiques, 1897).

Let such a line L be followed in a determinate direction. At any point on it, we know from (b) whether a trajectory crosses it from left to right or from right to left; and this

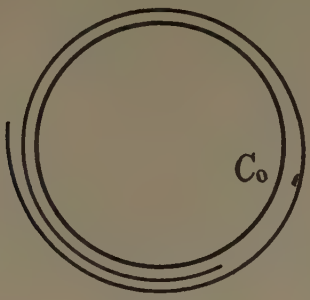

FIG. 2

sense of crossing can change only where L has a "contact," that is, at a point where the corresponding direction (X, Y) is touched by L (or at a multiple point of L). As such

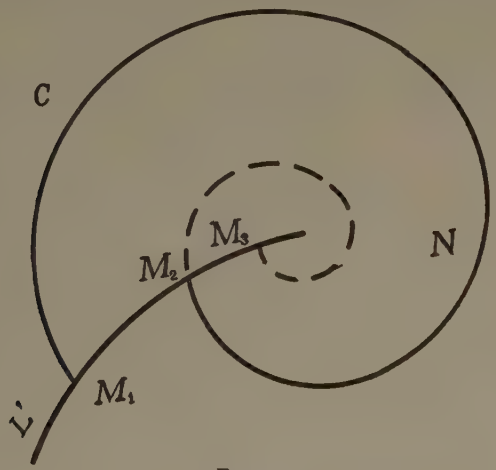

FIG. 3

contacts or multiple points are necessarily finite in number, we may assert that there exists, on L, at least one continuous arc L' which is crossed an infinity of times by the

same trajectory C, and this, in the same sense, say to the right.

Let M_1, M_2 be two such consecutive crossing points (fig. 3), in which case Poincaré says M_2 is the "consequent" of M_1. As we are on a surface of genus zero, and here this hypothesis appears for the first time in the argument, the arc M_1M_2 of L', together with the arc M_1NM_2 of C, will constitute the boundary of a certain area S, in which the trajectory is entering when it crosses L' at M_2: after which *it becomes impossible that C may ever go out of S.* For, and here we have the intervention of (a), this could not take place along M_1NM_2, nor could it either along L', since C would then cross L' from right to left, which is contrary to our hypothesis.

If so, the consequent M_3 of M_2 (fig. 3) must lie within S, therefore beyond M_2; and similarly, if we mark the successive points of intersection of L' with C, each of these must lie further on L' than the preceding one. On the other hand, the successive points cannot go indefinitely far along L', inasmuch as they must remain inside S: therefore, they must approach a common limiting position M_0.

M_0 is its own consequent, so that the trajectory through it, C_0, is closed.

C_0 is a limiting cycle for our original trajectory C, as the successive points M_1 of C indefinitely approach M_0. Thus, the announced conclusion is proved.

We see that we have obtained it by applying (a), that is to say, by the consideration of relations of a certain sort between the different trajectories: with the single difference, in this case, that the several trajectories considered are none other than the successive arcs of the same one.

Moreover, it is evident that the above facts are to a certain extent analogous to those which occurred in the

earlier consideration of a focus; and, indeed, it is clear that a focus may be considered as a subcase of a limiting cycle, the cycle being reduced to one point. But the existence and position of the focus is known in advance, while, on the contrary, there is no *a priori* information of the existence or position of a limiting cycle.

This, again, shows us the profound difference between the general case and the elementarily integrable equations. In these, of course, we can arrange for the introduction of limiting cycles. We find them if the differential equation, transformed with the use of polar coördinates ρ, ω, assumes the form

$$\frac{d\rho}{d\omega} = f(\rho).$$

Then every root ρ_0 of the equation $f(\rho) = 0$, the function f behaving regularly in the neighborhood of $\rho = \rho_0$, is the radius of a circle to which trajectories are asymptotic, in other terms, which is a limiting cycle. Perhaps it thus appears that this is not an exceptional case, now that we have been told so; but no geometer would have thought so before Poincaré's Memoir.

Moreover, the above case in which the very choice of coördinates points out the limiting cycle, still gives an incomplete, if not false, idea of the general case, in which, though the difficulty has certainly diminished by Poincaré's discovery, the difficulty in the research of limiting cycles seems to be comparable with the very integration of the equation itself.

This difficulty is the one which Poincaré now attacks. He cannot properly solve it by actually constructing the limiting cycles: he can only give hints on where they may be and where they cannot. The problem of course is to make these hints more and more precise; and this is indeed

one of the subjects where further researches would be the most interesting and, though difficult, could prove the most fruitful.

The method again rests on the aforementioned principle (b), that is, on the consideration of the sense of crossing between trajectories and an arbitrarily drawn line L; and, more precisely, on the consideration of such closed lines L which are "contactless cycles," that is, along which this sense can never change.

We immediately conceive that such a contactless cycle can have no point in common with any closed trajectory—therefore, with any limiting cycle: for a trajectory which has crossed it a first time, say from left to right, cannot cross it again from right to left. Thus, contactless cycles already give us places where limiting cycles cannot be.

Without explaining how they can also eventually show us regions where limiting cycles *must* be, we may add that the considerations by which Poincaré proved the exceptional character of centres (see above) are exactly of the same kind. Around a centre O, there must evidently exist a univocal integral $F(x, y)$, so that $F(x, y) = \text{const.}$ gives the successive closed trajectories around O. The direct calculation of such a function F, if existent, can be carried out by introducing polar coördinates, and leads to successive quadratures of trigonometric sums S_n with respect to the angular coördinate. The aforesaid conditions are that each S_n be without a constant term; and if any of them fails to be satisfied, this means the existence, around O, of a family of contactless cycles, which is exclusive of the possibility of any closed trajectory.

But this theory of centres already belongs to the third Memoir on Curves defined by differential equations, which we do not intend to examine here, as it would lead us into

the second period of the scientific life of Poincaré. Though the most important principle introduced in the last two Memoirs on that subject, the notion of integral invariants, again derives from the directing idea which we have already indicated,[1] namely, the simultaneous consideration of different trajectories and their mutual relations, and, though, also, the Memoirs on residues of double integrals, which appeared approximately at the same time, may be considered as the continuation of the previous research concerning meromorphic functions of two variables, these two new discoveries denote new directions in Poincaré's thought. The former, indeed, is nearer to the Memoir which won the prize offered by the King of Sweden and to the "Méthodes nouvelles de la Mécanique Céleste" than to the first two Memoirs in the Journal de Mathématiques, while the capital researches on the forms of planets, though still in the realm of Celestial Mechanics, may be regarded as the first stage of this genial thinker's work in the domain of Mathematical Physics.

J. HADAMARD.

[1] This general work of Poincaré on the qualitative theory of differential equations, which may almost be said to be the commencement of a new branch of mathematics, has been followed by several Memoirs, though certainly their number does not yet correspond to the interest and importance of the subject (which is to be explained by its great difficulty). We cannot omit pointing out, above all, Birkhoff's capital papers, not only on the proof of Poincaré's last theorem (Transactions of the American Mathematical Society, vol. XIV, p. 14-22; 1913) but on the intervention of "recurrent motions," on the application of the properties of point transformations (in the Acta Mathematica, vol. XLIII), or the study of periodic motions (in the Circolo Matematico). Among other geometers who have obtained valuable results by the application of Poincaré's principles, we have already mentioned M. Chazy, who has used them for a discussion of the problem of three bodies and has succeeded in giving an already precise classification of possible trajectories.

Nevertheless, in the immense field opened by Poincaré, many further researches—especially in such directions as Birkhoff's and Chazy's—would be of great interest.

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